

NORMS OF IDEMPOTENT SCHUR MULTIPLIERS

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ABSTRACT. Let \mathcal{D} be a masa in $\mathcal{B}(\mathcal{H})$ where \mathcal{H} is a separable Hilbert space. We find real numbers $\eta_0 < \eta_1 < \eta_2 < \cdots < \eta_6$ so that for every bounded, normal \mathcal{D} -bimodule map Φ on $\mathcal{B}(\mathcal{H})$ either $\|\Phi\| > \eta_6$, or $\|\Phi\| = \eta_k$ for some $k \in \{0, 1, 2, 3, 4, 5, 6\}$. When \mathcal{D} is totally atomic, these maps are the idempotent Schur multipliers and we characterise those with norm η_k for $0 \leq k \leq 6$. We also show that the Schur idempotents which keep only the diagonal and superdiagonal of an $n \times n$ matrix, or of an $n \times (n+1)$ matrix, both have norm $\frac{2}{n+1} \cot(\frac{\pi}{2(n+1)})$, and we consider the average norm of a random idempotent Schur multiplier as a function of dimension. Many of our arguments are framed in the combinatorial language of bipartite graphs.

Keywords: idempotent Schur multiplier, normal masa bimodule map, Hadamard product, norm, bipartite graph

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1. INTRODUCTION

Let \mathbb{F} be either \mathbb{R} or \mathbb{C} , and let $m, n \in \mathbb{N} \cup \{\aleph_0\}$. If $A = [a_{ij}]$ and $X = [x_{ij}]$ are $m \times n$ matrices with entries in \mathbb{F} , then the Schur product of A and X is their entrywise product:

$$A \bullet X = [a_{ij}x_{ij}].$$

This is also known as the Hadamard product. Let $\mathcal{B} = \mathcal{B}(\ell_n^2, \ell_m^2)$ be the space of matrices defining bounded linear operators $\ell_n^2 \rightarrow \ell_m^2$, where ℓ_k^2 is the k -dimensional Hilbert space of square-summable \mathbb{F} -valued sequences. An $m \times n$ matrix A with entries in \mathbb{F} is called a Schur multiplier if $X \mapsto A \bullet X$ leaves \mathcal{B} invariant. It then follows that Schur multiplication by A defines a bounded linear map $\mathcal{B} \rightarrow \mathcal{B}$, so the Schur norm of A given by

$$\|A\|_{\bullet} = \sup\{\|A \bullet X\|_{\mathcal{B}} : X \in \mathcal{B}, \|X\|_{\mathcal{B}} \leq 1\}$$

is finite. Under matrix addition, the Schur product \bullet and the norm $\|\cdot\|_{\bullet}$, the set of all $m \times n$ Schur multipliers forms a unital commutative semisimple Banach algebra. Several properties of Schur multipliers and the norm $\|\cdot\|_{\bullet}$ are known; see for example [2, 13, 6]. Here, we focus on the norms of the idempotent elements of this algebra: those Schur multipliers A for which every entry of A is either 0 or 1.

If $S \subseteq \mathbb{F}$, then we write $M_{m,n}(S)$ for the set of all $m \times n$ matrices with entries in S . For $m, n \in \mathbb{N}$, consider the finite sets of non-negative real numbers

$$\mathcal{N}(m, n) = \{\|A\|_{\bullet} : A \in M_{m,n}(\{0, 1\})\}.$$

We will see in Remark 3.4 below that this set does not depend on whether $\mathbb{F} = \mathbb{R}$ or $\mathbb{F} = \mathbb{C}$. Adding rows or columns of zeros to a matrix does not change its Schur norm, so if $n \leq n'$ and $m \leq m'$, then $\mathcal{N}(m, n) \subseteq \mathcal{N}(m', n')$. We will be interested in the set

$$\mathcal{N} = \mathcal{N}(\aleph_0, \aleph_0)$$

consisting of the norms of all idempotent Schur multipliers on $\mathcal{B}(\ell^2)$. Every element of \mathcal{N} is the supremum of a sequence in $\bigcup_{m,n \in \mathbb{N}} \mathcal{N}(m, n)$, obtained by considering the Schur norms of the upper-left hand corners of the corresponding infinite 0–1 matrix.

It has been known for some time that \mathcal{N} is closed under multiplication (consider $A_1 \otimes A_2$) and under suprema (consider $\bigoplus_i A_i$), that \mathcal{N} is not bounded above [12] and that \mathcal{N} contains accumulation points [3]. On the other hand, many basic properties of \mathcal{N} seem to be unknown. For example: is \mathcal{N} closed? Does \mathcal{N} have non-empty interior? Might we have $\mathcal{N} \supseteq [a, \infty)$ for some $a \geq 0$? Or, in the opposite direction, is \mathcal{N} actually countable?

We say that a non-empty open interval (a, b) is a gap in \mathcal{N} if $a, b \in \mathcal{N}$ but $(a, b) \cap \mathcal{N} = \emptyset$. The idempotent elements p of any Banach algebra satisfy

$$\|p\| = \|p^2\| \leq \|p\|^2,$$

so if $\|p\| \leq 1$ then $\|p\| \in \{0, 1\}$. In particular, this shows that $(0, 1)$ is a gap in \mathcal{N} . However, \mathcal{N} contains further gaps, a perhaps unexpected phenomenon. Indeed, Livschits [14] proves that

$$\{0, 1, \sqrt{4/3}\} \subseteq \mathcal{N} \subseteq \{0, 1\} \cup [\sqrt{4/3}, \infty),$$

so the open interval $(1, \sqrt{4/3})$ is also a gap in \mathcal{N} . Livschits' theorem has since been generalised by Katavolos and Paulsen [10], and has been recently used by Forrest and Runde to describe certain ideals of the Fourier algebra of a locally compact group [8].

We will show that there are at least four further gaps:

Theorem 1.1. *Consider the real numbers $\eta_0 < \eta_1 < \eta_2 < \eta_3 < \eta_4 < \eta_5 < \eta_6$ given by*

$$\begin{aligned} \eta_0 &= 0, & \eta_1 &= 1, & \eta_2 &= \sqrt{\frac{4}{3}}, & \eta_3 &= \frac{1 + \sqrt{2}}{2}, \\ \eta_4 &= \frac{1}{15} \sqrt{169 + 38\sqrt{19}}, & \eta_5 &= \sqrt{\frac{3}{2}}, & \eta_6 &= \frac{2}{5} \sqrt{5 + 2\sqrt{5}}. \end{aligned}$$

We have

$$\{\eta_0, \eta_1, \eta_2, \eta_3, \eta_4, \eta_5, \eta_6\} \subseteq \mathcal{N} \subseteq \{\eta_0, \eta_1, \eta_2, \eta_3, \eta_4, \eta_5\} \cup [\eta_6, \infty),$$

so (η_{j-1}, η_j) is a gap in \mathcal{N} for $1 \leq j \leq 6$.

Since it is fundamental to many of the calculations that follow, we recall here the connection between the problem of finding $\|A\|_\bullet$ and factorisations $A = S^*R$. If $m, n \in \mathbb{N}$ and $A \in M_{m,n}(\mathbb{C})$, the well-known Haagerup estimate states

$$\|A\|_\bullet \leq \|W\| \|V\| \quad \text{where} \quad A \bullet X = \sum_{j=1}^k W_j X V_j \quad \text{for all } X \in M_{m,n}(\mathbb{C}).$$

Here k is a natural number, W is a block row of $m \times m$ matrices W_1, W_2, \dots, W_k and V is a block column of $n \times n$ matrices V_1, V_2, \dots, V_k ; the norms of V and W are computed by allowing them to act as linear operators between Hilbert spaces of the appropriate finite dimensions. Moreover, the norm $\|A\|_\bullet$ is the minimum of these estimates $\|W\| \|V\|$. Stated in this generality, the same is true for an arbitrary elementary operator on $M_{m,n}(\mathbb{C})$; for Schur multipliers, the minimum is attained by a row W and a column V with $k \leq \min\{m, n\}$ for which the entries of W and V are all *diagonal* matrices. We can then rewrite the Haagerup estimate in the compact form

$$\|A\|_\bullet \leq c(S)c(R) \quad \text{where} \quad A = S^*R$$

by taking R to be the $k \times n$ matrix whose rows are the diagonals of the entries of V , and S to be the $k \times m$ matrix whose rows are the complex conjugates of the diagonals of the entries of W , and defining $c(R)$ and $c(S)$ to be the maximum of the ℓ^2 -norms of the columns of the corresponding matrices R and S . This notation comes from [1, 4].

The structure of this paper is as follows. We will use the combinatorial language of bipartite graphs to describe idempotent Schur multipliers, and this is explained in Section 2. Section 3 briefly recalls some basic results about the norms of general Schur multipliers, and casts them in this language. Section 4 is concerned with the calculation of the norms of the idempotent Schur multipliers corresponding to simple paths; these are the maps which keep only the main diagonal and superdiagonal elements of a matrix. Somewhat unexpectedly, we get the same answer in the $n \times n$ and the $n \times (n+1)$ cases. In Section 5 we compute or estimate the norms of some “small” idempotent Schur multipliers. Section 6 uses these results and simple combinatorial arguments to characterise the Schur idempotents with norm η_k for $1 \leq k \leq 6$, and hence to prove Theorem 1.1. Using work of Katavolos and Paulsen [10], this allows us to show in Section 7 that these gaps persist in the set of norms of all bounded, normal, idempotent masa bimodule maps on $\mathcal{B}(\mathcal{H})$ where \mathcal{H} is a separable Hilbert space. Finally, in Section 8 we estimate the average Schur norm of a random Schur

idempotent, in which each entry is chosen independently to be 1 with probability p and 0 with probability $1 - p$.

2. BIPARTITE GRAPHS

Let $m, n \in \mathbb{N} \cup \{\aleph_0\}$, and consider an $m \times n$ matrix $A = [a_{ij}]$ where each $a_{ij} \in \{0, 1\}$. To A we associate an undirected countable bipartite graph $G = G(A)$, specified as follows. The vertex set $V(G)$ is the disjoint union of two sets, R and C , where $|R| = m$ and $|C| = n$. Fixing enumerations $R = \{r_1, r_2, \dots\}$ and $C = \{c_1, c_2, \dots\}$, we define the edge set of G to be

$$E(G) = \{(r_i, c_j) : a_{ij} = 1\}.$$

For example, if

$$A = \begin{bmatrix} 1 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix},$$

then the corresponding graph is

$$G(A) = \text{⋈⋈⋈}$$

where we have drawn the set of “row vertices” $R = \{r_1, r_2, r_3\}$ above the “column vertices” $\{c_1, c_2, c_3, c_4\}$. In general, G will be a bipartite graph with bipartition (R, C) , which simply means that no edges join an element of R to an element of C . We call such a graph an (R, C) -bipartite graph. Clearly the map $A \mapsto G(A)$ is a bijection from the set of all $m \times n$ matrices of 0s and 1s onto $\Gamma(R, C)$, the set of all (R, C) -bipartite graphs. We remark in passing that in the linear algebra and spectral graph theory literature, A is called the biadjacency matrix of $G(A)$.

We will write $A = M(G)$ to mean that $G = G(A)$, and we adopt the shorthand

$$\|G\| := \|M(G)\|_{\bullet}.$$

In particular, if R and C are countably infinite sets, then

$$\mathcal{N} = \{\|G\| : G \in \Gamma(R, C)\}.$$

More generally, if X and Y are any sets and $G \subseteq X \times Y$, then we may think of G as a bipartite graph whose vertex set $V(G)$ is the disjoint union of X and Y , and whose edge set is $E(G) = G$. We write $\Gamma(X, Y)$ for the power set of $X \times Y$, viewed as the collection of all such bipartite graphs.

If $G \in \Gamma(X, Y)$ and $G' \in \Gamma(X', Y')$, then we say that the graphs G and G' are isomorphic if there is an isomorphism of bipartite graphs from G to G' . This means that there is a bijection $\theta : V(G) \rightarrow V(G')$ which either maps X onto X' and Y onto Y' or maps X onto Y' and Y onto X' , so that θ induces a bijection from $E(G)$ onto $E(G')$. We

do not distinguish between isomorphic graphs, so for example we write $G = G'$ if G and G' are merely isomorphic.

If $G_0 \in \Gamma(X_0, Y_0)$ and $G \in \Gamma(X, Y)$, then G_0 is an induced subgraph of G if $X_0 \subseteq X$, $Y_0 \subseteq Y$ and for $x_0 \in X_0$ and $y_0 \in Y_0$ we have

$$(x_0, y_0) \in E(G_0) \iff (x_0, y_0) \in E(G).$$

In other words, $G_0 = G \cap (X_0 \times Y_0)$; we will abbreviate this as $G_0 = G[X_0, Y_0]$.

If we merely have

$$(x_0, y_0) \in E(G_0) \implies (x_0, y_0) \in E(G),$$

so that G_0 may be obtained by removing some edges from an induced subgraph of G , then we say that G_0 is a subgraph of G . We will write $G_0 \leq G$ or $G \geq G_0$ to mean that G_0 (or a graph isomorphic to G_0) is an induced subgraph of G ; and we will write $G_0 \subseteq G$ to mean that G_0 (or a graph isomorphic to G_0) is a subgraph of G . Similarly, we write $G_0 < G$ to mean that $G_0 \leq G$ but G_0 is not isomorphic to G .

Definition 2.1. Let G be a graph and let v be a vertex of G . The set $N(v)$ of neighbours of v in G consists of all vertices joined to v by an edge of G . The degree $\deg(v)$ of v in G is the cardinality of $N(v)$. If the vertices of G have bounded degree, then we write

$$\deg(G) = \max_{v \in V(G)} \deg(v).$$

We say that vertices v, w in G are twins in G if $N(v) = N(w)$. A graph G is twin-free if no pair of distinct vertices are twins.

Proposition 2.2. *Any graph G has a maximal twin-free induced subgraph $\text{tf}(G)$, which is unique up to graph isomorphism. If G is bipartite, then so is $\text{tf}(G)$.*

Proof. Being twins is an equivalence relation on the vertices of G . If we choose a complete set of equivalence class representatives, then the corresponding induced subgraph of G is twin-free, and by construction it is maximal with respect to \leq among the twin-free induced subgraphs of G . Passing from one choice of equivalence class representatives to another produces an isomorphism of graphs. On the other hand, if v and w are any two distinct vertices in a twin-free induced subgraph $S \leq G$, then v and w are not twins in S , so they cannot be twins in G . So the vertices of S all lie in different equivalence classes, so S is an induced subgraph of one of the maximal induced subgraphs we have described. Since any subgraph of a bipartite graph is bipartite, the second assertion is trivial. \square

Remark 2.3. Note that $M(\text{tf}(G))$ is obtained from $M(G)$ by repeatedly deleting duplicate rows and columns.

Let G be any graph. If v, v' are distinct vertices of G , then a path in G from v to v' of length k is a finite sequence $(v_0, v_1, v_2, \dots, v_k)$ of vertices of G , where $v = v_0$ and $v' = v_k$, so that v_j is joined by an edge in G to v_{j+1} for $1 \leq j < k$. This is a simple path if no vertex appears twice. The distance between v and v' is the smallest possible length of such a path in G . Being joined by some path in G is an equivalence relation on the vertices of G ; by a connected component of G we mean an equivalence class for this relation, and we say that G is connected if it is a connected component of itself.

It is easy to see that:

Lemma 2.4. *A graph G is connected if and only if $\text{tf}(G)$ is connected.* \square

The size $|G|$ of a graph G is the cardinality of its vertex set. We say that G is finite if $|G| < \infty$. Let $\mathcal{F}(G)$ be the set

$$\mathcal{F}(G) = \{F \leq G : F \text{ is finite, connected and twin-free}\}.$$

We will use the following observation in Section 7.

Lemma 2.5. *Let X, Y be sets and let $G \in \Gamma(X, Y)$ be a connected bipartite graph. If $\mathcal{F}(G)$ contains finitely many non-isomorphic bipartite graphs, then $\text{tf}(G)$ is finite.*

Proof. Suppose instead that $\text{tf}(G) = G[S, T]$ where $S \subseteq X$ and $T \subseteq Y$ and S is infinite. Let A be a finite subset of S with $|A| > |F|$ for every $F \in \mathcal{F}(G)$. Since $G[S, T]$ is twin-free, for any pair a_1, a_2 of distinct vertices in A there is a vertex $t = t(a_1, a_2) \in T$ so that one of (a_1, t) and (a_2, t) is an edge of G , and the other is not. Consider

$$B = \{t(a_1, a_2) : a_1, a_2 \in A, a_1 \neq a_2\}.$$

Since $\text{tf}(G)$ is connected, we can find finite sets A', B' with $A \subseteq A' \subseteq S$ and $B \subseteq B' \subseteq T$ so that $G[A', B']$ is connected. Consider $F = \text{tf}(G[A', B'])$. By construction, $F \in \mathcal{F}(G)$. However, $|F| \geq |A|$ since no two vertices in A are twins in $G[A', B']$, so F cannot be (isomorphic to) an element of $\mathcal{F}(G)$, a contradiction. \square

3. BASIC RESULTS

If A and B are matrices, then we will write

$$A \simeq B$$

to mean that $B = UAV$ for some permutation matrices U, V ; in other words, permuting the rows and columns of A yields B .

The following facts about the norms of Schur multipliers are well-known.

Proposition 3.1. *Let A and B be matrices with countably many rows and columns.*

- (1) $\|A\|_{\bullet} = \|A^t\|_{\bullet}$.
- (2) If $A \simeq B$, then $\|A\|_{\bullet} = \|B\|_{\bullet}$.
- (3) If B can be obtained from A by deleting some rows or columns, then $\|B\|_{\bullet} \leq \|A\|_{\bullet}$.
- (4) $\|A_1 \oplus A_2 \oplus A_3 \oplus \dots\|_{\bullet} = \sup_j \|A_j\|_{\bullet}$.
- (5) $\|A\|_{\bullet} = \left\| \begin{bmatrix} A & A \\ A & A \end{bmatrix} \right\|_{\bullet}$.
- (6) If B can be obtained from A by duplicating rows or columns, then $\|B\|_{\bullet} = \|A\|_{\bullet}$.

Proof. Statements (1)–(4) all follow easily from properties of the operator norm $\|\cdot\|_{\mathcal{B}}$. For (5), let us write $S_A: \mathcal{B} \rightarrow \mathcal{B}$, $X \mapsto A \bullet X$ for the mapping of Schur multiplication by A . The two-fold ampliation $S_A^{(2)}: M_2(\mathcal{B}) \rightarrow M_2(\mathcal{B})$ of S_A (in the sense of operator space theory) may be naturally identified with S_B where $B = \begin{bmatrix} A & A \\ A & A \end{bmatrix}$. Now

$$\|A\|_{\bullet} = \|S_A\| \leq \|S_A^{(2)}\| = \|B\|_{\bullet} \leq \|S_A\|_{cb} = \|S_A\| = \|A\|_{\bullet}$$

where the equality $\|S_A\|_{cb} = \|S_A\|$ is a theorem commonly attributed to an unpublished manuscript of Haagerup (see [15, p. 115], for example) and is also established in [22]. We therefore have equality, hence (5). Replacing the number 2 in this argument with some other countable cardinal and using statement (3) then yields a proof of statement (6). \square

Specialising to idempotent Schur multipliers and restating in terms of bipartite graphs, we have:

Proposition 3.2. *Let R and C be countable sets, and let $G \in \Gamma(R, C)$.*

- (1) *If $G' \in \Gamma(R', C')$ and G' is isomorphic to G , then $\|G'\| = \|G\|$.*
- (2) *If $G_0 \leq G$ then $\|G_0\| \leq \|G\|$.*
- (3) *The norm of G is the supremum of the norms of the connected components of G .*
- (4) $\|G\| = \|\text{tf}(G)\|$.

Proof. (1) follows from assertions (1) and (2) of Proposition 3.1. For $j = 2, 3$, assertion (j) here is a rewording of assertion (j + 1) of Proposition 3.1. (4) follows easily using the proof of Proposition 2.2 and Proposition 3.1(6). \square

Remark 3.3. It is natural to ask whether Proposition 3.2(2) generalises to all subgraphs, and not merely induced subgraphs. In other words, is following implication valid?

$$G_0 \subseteq G \stackrel{?}{\implies} \|G_0\| \leq \|G\|$$

The answer is no. The complete graph K in $\Gamma(\aleph_0, \aleph_0)$ corresponds to the matrix of all 1s, which has Schur multiplier norm 1 since it gives the identity mapping, but as is well-known [12], the upper-triangular

subgraph $T \subseteq K$ whose matrix is

$$M(T) = \begin{bmatrix} 1 & 1 & 1 & \dots \\ 0 & 1 & 1 & \dots \\ 0 & 0 & 1 & \dots \\ \vdots & \vdots & \vdots & \ddots \end{bmatrix}$$

has $\|T\| = \infty$. Note that T is twin-free, but K is certainly not. In view of Proposition 3.2(4), we might then ask whether this implication holds provided either G alone, or both G and G_0 , are required to be twin-free. Again, the answer is no; a counterexample is given by (7) and (8) of Proposition 5.1 below.

Remark 3.4. We now explain why our results are identical regardless of whether we choose $\mathbb{F} = \mathbb{R}$ or $\mathbb{F} = \mathbb{C}$. Let $\mathcal{B}_{\mathbb{F}}$ be the space of bounded linear maps from $\ell_{n,\mathbb{F}}^2$ to $\ell_{m,\mathbb{F}}^2$, the corresponding ℓ^2 spaces with entries in \mathbb{F} . For a Schur multiplier $A \in M_{m,n}(\mathbb{F})$, we temporarily write $\|A\|_{\bullet,\mathbb{F}}$ for the norm of the map

$$S_{A,\mathbb{F}}: \mathcal{B}_{\mathbb{F}} \rightarrow \mathcal{B}_{\mathbb{F}}, \quad B \mapsto A \bullet B$$

and let $\|A\|_{\bullet,\mathbb{F},cb}$ be the completely bounded norm of $S_{A,\mathbb{F}}$. For the theory of operator spaces over \mathbb{R} , we refer to [19, 20]. Note that $S_{A,\mathbb{F}}$ is a bimodule map over the algebra $\mathcal{D}_{\mathbb{F}}$ of diagonal operators in $\mathcal{B}_{\mathbb{F}}$, and $\mathcal{D}_{\mathbb{F}}$ has a cyclic vector. Since the proof of [22, Theorem 2.1] works equally well in the real and complex cases, we have

$$\|A\|_{\bullet,\mathbb{F}} = \|A\|_{\bullet,\mathbb{F},cb}.$$

If A is a Schur multiplier in $M_{m,n}(\mathbb{R})$, then $S_{A,\mathbb{C}}$ is the complexification of $S_{A,\mathbb{R}}$. Hence by [20, Theorem 2.1],

$$\|A\|_{\bullet,\mathbb{R}} = \|A\|_{\bullet,\mathbb{R},cb} = \|A\|_{\bullet,\mathbb{C},cb} = \|A\|_{\bullet,\mathbb{C}}.$$

We conclude that a Schur multiplier with real entries has the same norm whether we consider it as a mapping on $\mathcal{B}_{\mathbb{R}}$ or on $\mathcal{B}_{\mathbb{C}}$.

For $\mathbb{F} = \mathbb{C}$, this allows a slight simplification of the formula defining the norm of a Schur multiplier with real entries. Indeed, any Schur multiplier $A \in M_{m,n}(\mathbb{R})$ has

$$\|A\|_{\bullet} = \sup_{X \in O(m,n)} \|A \bullet X\|_{\mathcal{B}_{\mathbb{R}}}$$

where $O(m,n)$ is the set of extreme points of the unit ball of $\mathcal{B}_{\mathbb{R}}$: the set of isometries in $\mathcal{B}_{\mathbb{R}}$ if $n \leq m$, or the set of coisometries if $n \geq m$.

4. NORMS OF SIMPLE PATHS

Definition 4.1. For $n \in \mathbb{N}$, the n -cycle $\Lambda(n)$ is the maximal cycle in $\Gamma(n,n)$; equivalently, it is connected and every vertex has degree 2. For example, $\Lambda(3) = \bowtie$.

The (n, n) path $\Sigma(n, n)$ and the $(n, n + 1)$ path $\Sigma(n, n + 1)$ are the maximal simple paths in $\Gamma(n, n)$ and $\Gamma(n, n + 1)$, respectively; for example, $\Sigma(3, 3) = \mathbb{N}\mathbb{N}$ and $\Sigma(3, 4) = \mathbb{N}\mathbb{N}\mathbb{N}$.

For $n \in \mathbb{N}$, we write $\theta_n = \frac{\pi}{2n}$. By [6, Example 4.6], we have

$$(*) \quad \|\Lambda(n)\| = \begin{cases} \frac{2}{n} \cot \theta_n & \text{if } n \text{ is even,} \\ \frac{2}{n} \csc \theta_n & \text{if } n \text{ is odd.} \end{cases}$$

The main result of this section is:

Theorem 4.2. *For every $n \in \mathbb{N}$,*

$$\|\Sigma(n, n)\| = \|\Sigma(n, n + 1)\| = \frac{2}{n + 1} \cot \theta_{n+1}.$$

Before the proof, we make some remarks.

Remark 4.3. Observe that while $\Sigma(n, n) < \Sigma(n, n + 1) < \Lambda(n + 1)$, the norms of the first two graphs are equal for every n and all three have equal norm for odd n . However, these assertions do not follow from any of the easy observations of Proposition 3.2 since these graphs are all connected and twin-free.

Question 4.4. Is there a combinatorial characterisation of the connected twin-free bipartite graphs $G_0 < G$ with $\|G_0\| = \|G\|$?

Remark 4.5. Theorem 4.2 improves the following bounds of Popa [18]:

$$\frac{1}{n} \left(\csc \left(\frac{\pi}{4n + 2} \right) - 1 \right) \leq \|\Sigma(n, n)\| \leq \frac{2}{n + 1} \cot \theta_{n+1}.$$

She establishes the upper bound using results of Mathias [13], and the lower bound using some eigenvalue formulae due to Yueh [23].

The following corollary is also noted in [18]. Another proof can be found by applying a theorem of Bennett [2, Theorem 8.1] asserting that the norm of a Toeplitz Schur multiplier A is the total variation of the Borel measure μ on \mathbb{T} with $a_{i-j} = \hat{\mu}(i - j)$.

Corollary 4.6. *The infinite matrix A with $a_{ij} = 1$ if $j \in \{i, i + 1\}$ and $a_{ij} = 0$ otherwise has $\|A\|_{\bullet} = 4/\pi$.*

Proof. The Schur multiplier norm of A is the supremum of the Schur multiplier norms of its $n \times n$ upper left-hand corners A_n , and $G(A_n) = \Sigma(n, n)$. Hence

$$\|A\|_{\bullet} = \sup_{n \geq 1} \|A_n\|_{\bullet} = \sup_{n \geq 1} \frac{2}{n + 1} \cot \theta_{n+1} = \frac{4}{\pi}. \quad \square$$

Recall that \mathcal{N} denotes the set of norms of all (bounded) Schur idempotents.

Remark 4.7. \mathcal{N} is not discrete: its accumulation points include 2 by [3], $\sqrt{2}$ by Remark 5.3 below, and $4/\pi$ by Corollary 4.6. By Theorem 1.1, the infimum of the set of accumulation points of \mathcal{N} is in the interval $[\eta_6, 4/\pi]$.

Question 4.8. Is this infimum equal to $4/\pi$?

Question 4.9. Is \mathcal{N} closed? Does it have non-empty interior? Are there any limit points from above which are not limit points from below?

We turn now to the proof of Theorem 4.2, which will occupy us for the rest of this section. Fix $n \in \mathbb{N}$. For $j \in \mathbb{Z}$, write

$$\kappa(j) = \cos(j\theta_{n+1}) \quad \text{and} \quad \lambda(j) = \sin(j\theta_{n+1})$$

where as above, $\theta_{n+1} = \frac{\pi}{2(n+1)}$. Clearly, $\lambda(j) = 0 \iff j \in 2(n+1)\mathbb{Z}$. The following useful identity, valid for $N \in \mathbb{N}$, $f \in \{\kappa, \lambda\}$ and $a, d \in \mathbb{Z}$ with $\lambda(d) \neq 0$, is an immediate consequence of the formulae in [11].

$$(\star) \quad \sum_{j=0}^N f(a + 2dj) = \frac{\lambda((N+1)d)}{\lambda(d)} f(a + Nd)$$

Lemma 4.10. *Let $a \in \mathbb{Z}$ and let $f, g, h \in \{\pm\kappa, \pm\lambda\}$.*

(1) *If $m \in 2\mathbb{Z}$ and $|m| \leq 2n$, then*

$$\sum_{j=0}^{2n+1} (-1)^j f(a + mj) = 0.$$

(2) *If $s, t \in \mathbb{Z}$ with $\max\{|s|, |t|\} \leq n-1$ and $s \equiv t \pmod{2}$, then*

$$\sum_{j=0}^{2n+1} (-1)^j f(a + 2j)g(sj)h(tj) = 0 = \sum_{j=0}^{2n+1} (-1)^j g(sj)h(tj).$$

Proof. (1) We have

$$\sum_{j=0}^{2n+1} (-1)^j f(a + mj) = \sum_{j=0}^n f(a + 2mj) - \sum_{j=0}^n f(a + m + 2mj).$$

If $m = 0$ then this difference is clearly 0. If $m \neq 0$, then $|m| < 2(n+1)$ gives $\lambda(m) \neq 0$ and $\lambda((n+1)m) = 0$ since m is even, so by (\star) we have

$$\sum_{j=0}^n f(a + 2mj) = \sum_{j=0}^n f(a + m + 2mj) = 0.$$

(2) Using the product-to-sum trigonometric identities, we can write

$$f(a + 2j)g(sj)h(tj) = \frac{1}{4} \sum_{k=0}^3 f_k(a + m_k j)$$

where $f_k \in \{\pm\kappa, \pm\lambda\}$ and

$$m_k = 2 + (-1)^k s + (-1)^{\lfloor k/2 \rfloor} t.$$

Since m_k is even and $|m_k| \leq 2 + |s| + |t| \leq 2n$, the first equality follows from (1). The second equality is proven using a simplification of the same argument. \square

Let ρ be the 2×2 rotation matrix

$$\rho = \begin{bmatrix} \kappa(1) & -\lambda(1) \\ \lambda(1) & \kappa(1) \end{bmatrix}.$$

Note that for $s \in \mathbb{Z}$, we have

$$\rho^s = \begin{bmatrix} \kappa(s) & -\lambda(s) \\ \lambda(s) & \kappa(s) \end{bmatrix}$$

so that, in particular, each entry of ρ^s is of the form $g(s)$ for some $g \in \{\kappa, \pm\lambda\}$. Define an $n \times n$ orthogonal matrix W by

$$W = \begin{cases} \rho \oplus \rho^3 \oplus \rho^5 \oplus \cdots \oplus \rho^{n-1} & \text{if } n \text{ is even,} \\ [1] \oplus \rho^2 \oplus \rho^4 \oplus \cdots \oplus \rho^{n-1} & \text{if } n \text{ is odd.} \end{cases}$$

Here, $[1]$ is the 1×1 matrix whose entry is 1 and \oplus is the block-diagonal direct sum of matrices. Let v be the $n \times 1$ vector

$$v = \begin{cases} [1 \ 0 \ 1 \ 0 \ \dots \ 1 \ 0]^* & \text{if } n \text{ is even,} \\ [1 \ 1 \ 0 \ 1 \ 0 \ \dots \ 1 \ 0]^* & \text{if } n \text{ is odd.} \end{cases}$$

For $j \in \mathbb{Z}$, let $q_j = W^j v$, and consider the rank one operators

$$Q_j = q_j q_j^*.$$

We write $\text{conv}(S)$ for the convex hull of a subset S of a vector space.

Proposition 4.11. *Consider the real numbers $t_j = \kappa(1) - \kappa(3 + 4j)$ for $0 \leq j \leq n$.*

(1) $t_j > 0$ for $0 \leq j < n$ and $t_n = 0$.

(2) $\sum_{j=0}^{2n+1} (-1)^j \kappa(a + 2j) Q_j = 0 = \sum_{j=0}^{2n+1} (-1)^j Q_j$ for any $a \in \mathbb{R}$.

(3) $\sum_{j=0}^{n-1} t_j Q_{2j} = \sum_{j=0}^{n-1} t_j Q_{2(n-j)-1}$.

(4) $\text{conv}(\{Q_0, Q_2, \dots, Q_{2(n-1)}\}) \cap \text{conv}(\{Q_1, Q_3, \dots, Q_{2n-1}\}) \neq \emptyset$.

Proof. (1) This is clear.

(2) The k th entry of q_j has the form $g_k(s_k j)$ where $g_k \in \{\kappa, \pm\lambda\}$ and $s_k \in \mathbb{Z}$ with $|s_k| \leq n-1$ and $s_k \equiv n-1 \pmod{2}$. Hence the (k, ℓ) entry of Q_j is $g_k(s_k j) g_\ell(s_\ell j)$, so the claim follows from Lemma 4.10(2).

(3) For $\ell \in \mathbb{Z}$, we have $W^\ell Q_j W^{-\ell} = Q_{j+\ell}$. By (2),

$$W^{-1} \left(\sum_{j=0}^{2n+1} (-1)^j \kappa(1+2j) Q_j \right) W = \sum_{j=-1}^{2n} (-1)^{j+1} \kappa(3+2j) Q_j = 0.$$

Rearranging, reindexing and using the identity $\kappa(4(n+1)-x) = \kappa(x)$ gives

$$\sum_{j=0}^n \kappa(3+4j) Q_{2j} = \sum_{j=0}^n \kappa(3+4j) Q_{2(n-j)-1}.$$

We have $W^{2(n+1)} = (-1)^{n+1} I$, so

$$Q_{-1} = W^{2(n+1)} Q_{-1} W^{-2(n+1)} = Q_{2n+1}.$$

By the second equality in (2),

$$\sum_{j=0}^n \kappa(1) Q_{2j} = \sum_{j=0}^n \kappa(1) Q_{2(n-j)-1}.$$

Taking differences gives

$$\sum_{j=0}^n t_j Q_{2j} = \sum_{j=0}^n t_j Q_{2(n-j)-1}.$$

Since $t_n = 0$, this is the desired identity.

(4) This is immediate from (1) and (3). □

For $1 \leq j \leq n-1$, let

$$r_j = \begin{cases} \sqrt{\frac{2}{n+1}} & \text{if } j = 1 \text{ and } n \text{ is odd,} \\ \sqrt{\frac{4\kappa(j)}{n+1}} & \text{otherwise.} \end{cases}$$

Let r be the $n \times 1$ vector

$$r = \begin{cases} [r_1 \ 0 \ r_3 \ 0 \ \dots \ r_{n-1} \ 0]^* & \text{if } n \text{ is even,} \\ [r_1 \ r_2 \ 0 \ r_4 \ 0 \ \dots \ r_{n-1} \ 0]^* & \text{if } n \text{ is odd.} \end{cases}$$

A calculation using (★) gives

$$\|r\|^2 = \frac{2}{n+1} \cot \theta_{n+1}.$$

Consider the rank one operators

$$P_j = W^j r (W^j r)^*$$

for $j \in \mathbb{Z}$.

Remark 4.12. The diagonal matrix

$$D = \begin{cases} \text{diag}(r_1, r_1, r_3, r_3, \dots, r_{n-1}, r_{n-1}) & \text{if } n \text{ is even} \\ \text{diag}(r_1, r_2, r_2, r_4, r_4, \dots, r_{n-1}, r_{n-1}) & \text{if } n \text{ is odd} \end{cases}$$

commutes with W and $Dv = r$; hence $DQ_jD = P_j$. Since D is invertible, it follows that for any finite collection of scalars t_j we have

$$\sum_j t_j P_j = 0 \iff \sum_j t_j Q_j = 0.$$

Let $R = [r \ W^2r \ W^4r \ \dots \ W^{2(n-1)}r]$ and let $S = WR$. Also let $\tilde{R} = [R \ W^{2n}r]$ and let $\tilde{S} = W\tilde{R}$. Let us write X_i for the i th column of a matrix X . Since W is an isometry, for $1 \leq i, j \leq n+1$ we have

$$\|\tilde{R}_j\|^2 = \|\tilde{S}_i\|^2 = \|r\|^2 = \frac{2}{n+1} \cot \theta_{n+1}.$$

Let \tilde{B} be the $(n+1) \times (n+1)$ matrix whose (i, j) entry is

$$b_{ij} = \begin{cases} 1 & \text{if } j-i \in \{0, 1\}, \\ (-1)^{n+1} & \text{if } (i, j) = (n+1, 1), \\ 0 & \text{otherwise.} \end{cases}$$

Let B be the upper-left $n \times n$ corner of \tilde{B} and let B' consist of the first n rows of \tilde{B} . Observe that $G(B) = \Sigma(n, n)$, $G(B') = \Sigma(n, n+1)$ and if n is odd, then $G(\tilde{B}) = \Lambda(n+1)$.

Proposition 4.13. *We have $S^*R = B$ and $\tilde{S}^*\tilde{R} = \tilde{B}$.*

Proof. Since S^*R is the upper-left $n \times n$ corner of $\tilde{S}^*\tilde{R}$, it suffices to show that $\tilde{S}^*\tilde{R} = \tilde{B}$. Let $k = |2(i-j)+1|$, a positive odd integer. We have

$$(\tilde{S}^*\tilde{R})_{i,j} = \langle \tilde{R}_j, \tilde{S}_i \rangle = \langle W^{2(j-1)}r, W^{2(i-1)+1}r \rangle = \langle W^k r, r \rangle.$$

Since $W^{2(n+1)} = (-1)^{n+1}I$, the $(n+1, 1)$ entry of $\tilde{S}^*\tilde{R}$ is

$$\langle W^{2n+1}r, r \rangle = (-1)^{n+1} \langle W^*r, r \rangle = (-1)^{n+1} \langle Wr, r \rangle.$$

It therefore only remains to show that

$$\langle W^k r, r \rangle = \begin{cases} 1 & \text{if } k = 1 \\ 0 & \text{if } k \text{ is odd with } 3 \leq k \leq 2n-1. \end{cases}$$

We prove this by direct calculation, giving the details for even n ; the calculation for odd n is very similar. We have

$$\begin{aligned}
\langle Wr, r \rangle &= \sum_{j=0}^{n/2-1} r_{1+2j}^2 \kappa(1+2j) \\
&= \frac{4}{n+1} \sum_{j=0}^{n/2-1} \kappa(1+2j)^2 \\
&= \frac{2}{n+1} \sum_{j=0}^{n/2-1} 1 + \kappa(2+4j) \\
&= \frac{2}{n+1} \left(\frac{n}{2} + \frac{1}{2} \right) = 1
\end{aligned}$$

Here we have used (\star) to perform the summation in the penultimate line. If $3 \leq k \leq 2n-1$ and k is odd, then

$$\begin{aligned}
\langle W^k r, r \rangle &= \sum_{j=0}^{n/2-1} r_{1+2j}^2 \kappa(k(1+2j)) \\
&= \frac{4}{n+1} \sum_{j=0}^{n/2-1} \kappa(1+2j) \kappa(k(1+2j)) \\
&= \frac{2}{n+1} \sum_{j=0}^{n/2-1} \kappa((k-1)(1+2j)) + \kappa((k+1)(1+2j)) \\
&= \frac{1}{n+1} \left(\frac{\lambda(n(k-1))}{\lambda(k-1)} + \frac{\lambda(n(k+1))}{\lambda(k+1)} \right).
\end{aligned}$$

Since k is odd, $(k \pm 1)(n+1)\theta_{n+1} = \frac{k \pm 1}{2}\pi$ is an integer multiple of π and

$$\lambda(n(k \pm 1)) = \lambda((k \pm 1)(n+1) - (k \pm 1)) = (-1)^{(k \mp 1)/2} \lambda(k \pm 1),$$

so

$$\langle W^k r, r \rangle = \frac{1}{n+1} ((-1)^{(k+1)/2} + (-1)^{(k-1)/2}) = 0. \quad \square$$

Proof of Theorem 4.2. By Proposition 4.11 and Remark 4.12, there are two sets of positive scalars $\{a_j\}_{j=1}^n$ and $\{b_j\}_{j=1}^n$, each summing to 1, so that

$$\sum_{j=1}^n a_j R_j R_j^* = \sum_{j=1}^n b_j S_j S_j^*.$$

The $n \times n$ diagonal matrices $X = \text{diag}(\sqrt{a_j})$ and $Y = \text{diag}(\sqrt{b_j})$ have

$$RX(RX)^* = SY(SY)^*,$$

so there is a unitary matrix U with $RX = SYU$. (Indeed, B and Y are both invertible, so SY is invertible and $U = RX(SY)^{-1}$ has real entries and is an orthogonal matrix). As shown in [1], this implies that the factorisation $B = S^*R$ attains the Haagerup bound. Indeed, the unit vectors $x = [\sqrt{a_j}]_{1 \leq j \leq n}$ and $y = [\sqrt{b_j}]_{1 \leq j \leq n}$ satisfy

$$\begin{aligned} \langle (B \bullet U^t)x, y \rangle &= \text{trace}((SY)^*RXU) = \text{trace}(S^*SY Y^*) \\ &= \sum_{j=1}^n \|S_j\|^2 |y_j|^2 = c(S)^2 = c(S)c(R), \end{aligned}$$

so by Proposition 4.13,

$$\begin{aligned} \frac{2}{n+1} \cot \theta_{n+1} &= c(S)c(R) \leq \|B \bullet U^t\| \\ &\leq \|B\|_{\bullet} \leq \|\tilde{B}\|_{\bullet} \leq c(\tilde{S})c(\tilde{R}) = \frac{2}{n+1} \cot \theta_{n+1}. \end{aligned}$$

Hence $\|\Sigma(n, n)\| = \|B\|_{\bullet} = \|\Sigma(n, n+1)\| = \|\tilde{B}\|_{\bullet} = \frac{2}{n+1} \cot \theta_{n+1}$. \square

5. CALCULATIONS AND ESTIMATES OF SMALL NORMS

In this section, we calculate or estimate the norms of some particular idempotent Schur multipliers. Our first result is Proposition 5.1, in which we find the exact norms of some idempotent Schur multipliers in low dimensions. We then find lower bounds for the norms of some other Schur idempotents which we will use to establish Theorem 1.1 in the following section.

Proposition 5.1. (1) $\|\mathbb{I}\| = \eta_1 = 1$.

(2) $\|\mathbb{N}\| = \|\mathbb{N}\|_{\bullet} = \eta_2 = \sqrt{4/3} \approx 1.15470$.

(3) $\|\mathbb{N}\mathbb{N}\| = \|\mathbb{N}\mathbb{N}\|_{\bullet} = \|\mathbb{N}\mathbb{N}\mathbb{N}\| = \eta_3 = \frac{1+\sqrt{2}}{2} \approx 1.20711$.

(4) $\|\mathbb{N}\mathbb{N}\mathbb{N}\| = \|\mathbb{N}\mathbb{N}\mathbb{N}\|_{\bullet} = \eta_4 = \frac{1}{15} \sqrt{169 + 38\sqrt{19}} \approx 1.21954$.

(5) $\|\mathbb{N}\mathbb{N}\mathbb{N}\mathbb{N}\| = \eta_5 = \sqrt{3/2} \approx 1.22474$.

(6) $\|\mathbb{N}\mathbb{N}\mathbb{N}\mathbb{N}\| = \|\mathbb{N}\mathbb{N}\mathbb{N}\mathbb{N}\|_{\bullet} = \eta_6 = \frac{2}{5} \sqrt{5 + 2\sqrt{5}} \approx 1.23107$.

(7) $\|\mathbb{N}\mathbb{N}\mathbb{N}\mathbb{N}\mathbb{N}\| = \frac{1}{15} (9 + 4\sqrt{6}) \approx 1.25320$.

(8) $\|\mathbb{N}\mathbb{N}\mathbb{N}\mathbb{N}\mathbb{N}\mathbb{N}\| = 9/7 \approx 1.28571$.

(9) $\|\mathbb{N}\mathbb{N}\mathbb{N}\mathbb{N}\mathbb{N}\mathbb{N}\mathbb{N}\| = 4/3 \approx 1.33333$.

Proof. (1) is trivial, and assertions (2), (3) and (6) are consequences of Theorem 4.2 and equation (*).

(4) Let $B = \begin{bmatrix} 1 & 0 & 0 & 1 & 0 \\ 1 & 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 1 \end{bmatrix} \simeq M(\mathbb{N}\mathbb{N}\mathbb{N})$. Consider the matrices

$$P = \begin{bmatrix} \eta & \alpha & \beta \\ \alpha & \eta & \alpha \\ \beta & \alpha & \eta \end{bmatrix} \quad \text{and} \quad Q = \begin{bmatrix} \eta & \gamma & \delta & \sigma & \tau \\ \gamma & \eta & \gamma & -\sigma & -\sigma \\ \delta & \gamma & \eta & \tau & \sigma \\ \sigma & -\sigma & \tau & 2\sigma & \alpha \\ \tau & -\sigma & \sigma & \alpha & 2\sigma \end{bmatrix}$$

where $\eta = \eta_4$ and

$$\begin{aligned}\alpha &= \frac{1}{15}\sqrt{139 - 22\sqrt{19}}, & \beta &= -\frac{1}{15}\sqrt{24 - 2\sqrt{19}}, \\ \gamma &= \frac{2}{15}\sqrt{16 + 2\sqrt{19}}, & \delta &= \frac{1}{15}\sqrt{424 - 82\sqrt{19}}, \\ \sigma &= \frac{1}{15}\sqrt{61 + 2\sqrt{19}}, & \tau &= -\frac{1}{15}\sqrt{256 - 58\sqrt{19}}.\end{aligned}$$

One can check with a computer algebra system that $C = \begin{bmatrix} P & B \\ B^* & Q \end{bmatrix}$ has rank 3 and its non-zero eigenvalues are positive, so C is positive semidefinite. The maximum diagonal entry of C is $\max\{\eta, 2\sigma\} = \eta$, so $\|\text{\textcircled{A}}\text{\textcircled{A}}\text{\textcircled{A}}\text{\textcircled{A}}\| \leq \eta$ by [16] (see also [15, Exercise 8.8(v)]).

On the other hand,

$$U = \frac{1}{15} \begin{bmatrix} 8 + \sqrt{19} & -\sqrt{74 - 2\sqrt{19}} & -7 + \sqrt{19} \\ \sqrt{74 - 2\sqrt{19}} & 1 + 2\sqrt{19} & \sqrt{74 - 2\sqrt{19}} \\ -7 + \sqrt{19} & -\sqrt{74 - 2\sqrt{19}} & 8 + \sqrt{19} \end{bmatrix}$$

is orthogonal, and if $B = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix} \simeq M(\text{\textcircled{A}}\text{\textcircled{A}})$, then $\|B \bullet U\| = \eta \leq \|\text{\textcircled{A}}\text{\textcircled{A}}\|$. Since $\text{\textcircled{A}}\text{\textcircled{A}} \leq \text{\textcircled{A}}\text{\textcircled{A}}\text{\textcircled{A}}\text{\textcircled{A}}$, this shows that $\eta \leq \|\text{\textcircled{A}}\text{\textcircled{A}}\| \leq \|\text{\textcircled{A}}\text{\textcircled{A}}\text{\textcircled{A}}\text{\textcircled{A}}\| \leq \eta$ and we have equality.

(5) Let

$$S = \frac{1}{2 \cdot 54^{1/4}} \begin{bmatrix} 2\sqrt{6} & 2\sqrt{6} & 2\sqrt{6} \\ -2\sqrt{3} & \sqrt{3} & \sqrt{3} \\ 0 & 3 & -3 \end{bmatrix} \quad \text{and} \quad R = \frac{1}{54^{1/4}} \begin{bmatrix} 3 & 1 & 1 & 1 \\ 0 & -2\sqrt{2} & \sqrt{2} & \sqrt{2} \\ 0 & 0 & \sqrt{6} & -\sqrt{6} \end{bmatrix}.$$

Then $S^*R = \begin{bmatrix} 1 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 \\ 1 & 0 & 0 & 1 \end{bmatrix} \simeq M(\text{\textcircled{A}}\text{\textcircled{A}}\text{\textcircled{A}})$, so $\|\text{\textcircled{A}}\text{\textcircled{A}}\text{\textcircled{A}}\| \leq c(S)c(R) = \sqrt{3/2}$. On the other hand, consider

$$V = \frac{1}{4} \begin{bmatrix} \sqrt{5} & 3 & -1 & -1 \\ \sqrt{5} & -1 & 3 & -1 \\ \sqrt{5} & -1 & -1 & 3 \end{bmatrix}.$$

It is easy to see that V is a coisometry with $\|\begin{bmatrix} 1 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 \\ 1 & 0 & 0 & 1 \end{bmatrix} \bullet V\| = \sqrt{3/2}$. Hence $\|\text{\textcircled{A}}\text{\textcircled{A}}\text{\textcircled{A}}\| \geq \sqrt{3/2}$.

(7) Consider

$$S = \begin{bmatrix} 1 & 1 & 1/2 \\ a & -a & b \\ -a & a & c \end{bmatrix} \quad \text{and} \quad R = \begin{bmatrix} 1 & 1 & 1/2 \\ a & -a & b \\ a & -a & -c \end{bmatrix}$$

where

$$a = \sqrt{\frac{1}{15}(-3 + 2\sqrt{6})}, \quad b = \frac{1}{2}\sqrt{\frac{1}{15}(3 + 8\sqrt{6})} \quad \text{and} \quad c = \sqrt{\frac{1}{30}(9 + 4\sqrt{6})}.$$

Since $a(b+c) = \frac{1}{2}$ and $b^2 - c^2 = -\frac{1}{4}$, we have $S^*R = \begin{bmatrix} \frac{1}{15} & \frac{1}{15} & \frac{1}{15} \\ \frac{1}{15} & \frac{1}{15} & \frac{1}{15} \\ \frac{1}{15} & \frac{1}{15} & \frac{1}{15} \end{bmatrix} \simeq M(\mathbb{M} \times \mathbb{M})$, so $\|\mathbb{M} \times \mathbb{M}\| \leq c(S)c(R) = \frac{1}{15}(9 + 4\sqrt{6})$. On the other hand, calculations may be performed to show that the matrix

$$U = \frac{1}{15} \begin{bmatrix} 9 - \sqrt{6} & \sqrt{54 - 6\sqrt{6}} & 2\sqrt{21 + 6\sqrt{6}} \\ \sqrt{54 - 6\sqrt{6}} & 3(1 + \sqrt{6}) & -2\sqrt{27 - 3\sqrt{6}} \\ 2\sqrt{21 + 6\sqrt{6}} & -2\sqrt{27 - 3\sqrt{6}} & 3 - 2\sqrt{6} \end{bmatrix}$$

is orthogonal, and $\left\| \begin{bmatrix} \frac{1}{15} & \frac{1}{15} & \frac{1}{15} \\ \frac{1}{15} & \frac{1}{15} & \frac{1}{15} \\ \frac{1}{15} & \frac{1}{15} & \frac{1}{15} \end{bmatrix} \bullet U \right\| = \frac{1}{15}(9 + 4\sqrt{6})$.

(8) Let

$$S = \frac{1}{\sqrt{14}} \begin{bmatrix} 3 & 4 & 3 \\ -\sqrt{2} & \sqrt{2} & -\sqrt{2} \\ -\sqrt{7} & 0 & \sqrt{7} \end{bmatrix} \quad \text{and} \quad R = \frac{1}{\sqrt{14}} \begin{bmatrix} 3 & 4 & 3 \\ \sqrt{2} & -\sqrt{2} & \sqrt{2} \\ -\sqrt{7} & 0 & \sqrt{7} \end{bmatrix}.$$

Then $S^*R = \begin{bmatrix} \frac{1}{15} & \frac{1}{15} & \frac{1}{15} \\ \frac{1}{15} & \frac{1}{15} & \frac{1}{15} \\ \frac{1}{15} & \frac{1}{15} & \frac{1}{15} \end{bmatrix} = M(\mathbb{M})$, so $\|\mathbb{M}\| \leq c(S)c(R) = 9/7$. The matrix

$$U = \frac{1}{7} \begin{bmatrix} 3 & 2\sqrt{6} & -4 \\ 2\sqrt{6} & 1 & 2\sqrt{6} \\ -4 & 2\sqrt{6} & 3 \end{bmatrix}$$

is orthogonal, and $\left\| \begin{bmatrix} \frac{1}{15} & \frac{1}{15} & \frac{1}{15} \\ \frac{1}{15} & \frac{1}{15} & \frac{1}{15} \\ \frac{1}{15} & \frac{1}{15} & \frac{1}{15} \end{bmatrix} \bullet U \right\| = 9/7 \leq \|\mathbb{M}\|$.

(9) This is proven in [3, Theorem 2.1], and is also a consequence of (*). \square

Remark 5.2. Since $M(\mathbb{M}) \simeq \begin{bmatrix} \frac{1}{15} & \frac{1}{15} & \frac{1}{15} \\ \frac{1}{15} & \frac{1}{15} & \frac{1}{15} \\ \frac{1}{15} & \frac{1}{15} & \frac{1}{15} \end{bmatrix}$, part (7) of the preceding result gives the norm of the upper-triangular truncation map on the 3×3 matrices. This result has previously been stated in [1, p. 131], but a detailed calculation does not appear in that reference.

Remark 5.3. Proposition 5.1(5) may be generalised to show that

$$\|[\mathbf{1} \ I_n]\|_{\bullet} = \sqrt{\frac{2n}{n+1}}$$

where $\mathbf{1}$ is the $n \times 1$ vector of all ones and I_n is the $n \times n$ identity matrix. We omit the details.

Proposition 5.4. $\|\mathbb{N} \times \mathbb{N}\| > \|\mathbb{M} \times \mathbb{M}\|$.

Proof. The matrix $U = \frac{1}{6} \begin{bmatrix} \frac{3}{2} & \frac{3}{2} & \frac{3}{2} & \frac{3}{2} \\ \frac{3}{2} & -\frac{5}{2} & \frac{1}{2} & \frac{1}{2} \\ \frac{3}{2} & \frac{1}{2} & -\frac{5}{2} & \frac{1}{2} \\ \frac{3}{2} & \frac{1}{2} & \frac{1}{2} & -\frac{5}{2} \end{bmatrix}$ has $U = 2P - I$ where P is the rank one projection onto the linear span of $\begin{bmatrix} \frac{3}{2} \\ \frac{1}{2} \\ \frac{1}{2} \\ \frac{1}{2} \end{bmatrix}$, so U is orthogonal.

Clearly $B = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$ has $M(\mathbb{I} \mathbb{N} \mathbb{N} \mathbb{I}) \simeq B$, and $Y = 6B \bullet U$ has

$$\|Y\|^2 = \left\| \begin{bmatrix} 3 & 3 & 3 & 3 \\ 0 & -5 & 0 & 0 \\ 0 & 0 & -5 & 0 \\ 0 & 0 & 0 & -5 \end{bmatrix} \begin{bmatrix} 3 & 0 & 0 & 0 \\ 0 & -5 & 0 & 0 \\ 3 & 0 & -5 & 0 \\ 3 & 0 & 0 & -5 \end{bmatrix} \right\| = \left\| 25I + \begin{bmatrix} 11 & -15 & -15 & -15 \\ -15 & 0 & 0 & 0 \\ -15 & 0 & 0 & 0 \\ -15 & 0 & 0 & 0 \end{bmatrix} \right\|.$$

Since the norm of YY^* is its spectral radius, a calculation gives

$$\|Y\|^2 = \frac{1}{2}(61 + \sqrt{2821}).$$

Hence

$$\|\mathbb{I} \mathbb{N} \mathbb{N} \mathbb{I}\| = \|B\|_{\bullet} \geq \frac{1}{6}\|Y\| = \frac{1}{6}\sqrt{\frac{1}{2}(61 + \sqrt{2821})} > \frac{1}{15}(9 + 4\sqrt{6}) = \|\mathbb{N} \mathbb{N} \mathbb{N}\|. \quad \square$$

Proposition 5.5. $\|\mathbb{I} \mathbb{N} \mathbb{N} \mathbb{I}\| > \|\mathbb{N} \mathbb{N} \mathbb{N}\|$.

Proof. Consider the unit vectors x and y appearing in the proof of Theorem 4.2 in the case $n = 4$. It turns out that

$$x = \frac{1}{\sqrt{5}} \begin{bmatrix} \sqrt{\frac{1}{2}(3 - \sqrt{5})} \\ \sqrt{\frac{1}{2}(1 + \sqrt{5})} \\ \sqrt{2} \\ 1 \end{bmatrix} \quad \text{and} \quad y = \frac{1}{\sqrt{5}} \begin{bmatrix} 1 \\ \sqrt{2} \\ \sqrt{\frac{1}{2}(1 + \sqrt{5})} \\ \sqrt{\frac{1}{2}(3 - \sqrt{5})} \end{bmatrix}$$

and that the matrix $B = \begin{bmatrix} 1 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 1 & 0 \end{bmatrix} \simeq M(\mathbb{I} \mathbb{N} \mathbb{N} \mathbb{I})$ has $\|B^t \bullet (xy^*)\|_1 > 1.235 > \|\mathbb{N} \mathbb{N} \mathbb{N}\|$, where $\|\cdot\|_1$ is the trace-class norm. It is well-known and easy to see that $S_B: \mathcal{B} \rightarrow \mathcal{B}$, $A \mapsto B \bullet A$ is the dual of $T_{B^t}: \mathcal{C}_1 \rightarrow \mathcal{C}_1$, $C \mapsto B^t \bullet C$, the mapping of Schur multiplication by B^t on the trace-class operators \mathcal{C}_1 (viewed as the predual of \mathcal{B}). Since x and y are unit vectors, $\|xy^*\|_1 = 1$ and so $\|B\|_{\bullet} = \|S_B\| = \|T_{B^t}\| > \|\mathbb{N} \mathbb{N} \mathbb{N}\|$. \square

Proposition 5.6. $\|\mathbb{I} \mathbb{N} \mathbb{N} \mathbb{I}\| > \|\mathbb{N} \mathbb{N} \mathbb{N}\|$.

Proof. Consider

$$B = \begin{bmatrix} 1 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 \\ 1 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 \end{bmatrix} \quad \text{and} \quad U = \frac{1}{4} \begin{bmatrix} \sqrt{5} & 3 & -1 & -1 \\ \sqrt{5} & -1 & 3 & -1 \\ \sqrt{5} & -1 & -1 & 3 \\ -1 & \sqrt{5} & \sqrt{5} & \sqrt{5} \end{bmatrix}$$

The matrix U is orthogonal, and $M(\mathbb{I} \mathbb{N} \mathbb{N} \mathbb{I}) \simeq B$. Now

$$16\|B \bullet U\|^2 = \left\| \begin{bmatrix} 15 & 3\sqrt{5} & 3\sqrt{5} & 3\sqrt{5} \\ 3\sqrt{5} & 9 & 0 & 0 \\ 3\sqrt{5} & 0 & 9 & 0 \\ 3\sqrt{5} & 0 & 0 & 14 \end{bmatrix} \right\| = \|9I + Z\|$$

where $Z = \begin{bmatrix} 6 & \sqrt{5} & \sqrt{5} & \sqrt{5} \\ \sqrt{5} & 0 & 0 & 0 \\ \sqrt{5} & 0 & 0 & 0 \\ \sqrt{5} & 0 & 0 & 5 \end{bmatrix}$, which has characteristic polynomial $p(x) = x(x^3 - 11x - 105x + 450)$. By estimating the roots of $p(x)$, one can

show that $\|B \bullet U\| = \frac{1}{4}\sqrt{9+\lambda}$ where λ is the largest root of $p(x)$, and that $\|\mathbb{I} \bowtie \mathbb{I}\| \geq \|B \bullet U\| > \|\mathbb{N} \mathbb{N} \mathbb{N}\|$. \square

Proposition 5.7. $\|\mathbb{N} \mathbb{N} \mathbb{N}\| > \|\mathbb{N} \mathbb{N} \mathbb{N}\|$.

Proof. Consider the symmetric matrices

$$B = \begin{bmatrix} 0 & 0 & 0 & 1 \\ 0 & 1 & 1 & 1 \\ 0 & 1 & 0 & 0 \\ 1 & 1 & 0 & 0 \end{bmatrix} \quad \text{and} \quad U = \begin{bmatrix} 0 & 0 & -1/\sqrt{2} & 1/\sqrt{2} \\ 0 & 1/3 & 2/3 & 2/3 \\ -1/\sqrt{2} & 2/3 & -1/6 & -1/6 \\ 1/\sqrt{2} & 2/3 & -1/6 & -1/6 \end{bmatrix}.$$

By direct calculation, U is orthogonal, and $M(\mathbb{N} \mathbb{N} \mathbb{N}) \simeq B$. The characteristic polynomial of $B \bullet U$ is $p(x) = \frac{1}{18}(x+1)(18x^3 - 24x^2 - x + 4)$. It is easy to see that $p(x)$ has two negative roots and two positive roots, and the smallest root is -1 while the largest root is larger than 1. Since $B \bullet U$ is symmetric, $\|B \bullet U\|$ is the spectral radius of $p(x)$, which is the largest root of $p(x)$. But $p(\|\mathbb{N} \mathbb{N} \mathbb{N}\|) < 0$ and $p'(x) > 0$ for $x > 1$, so $\|\mathbb{N} \mathbb{N} \mathbb{N}\| \geq \|B \bullet U\| > \|\mathbb{N} \mathbb{N} \mathbb{N}\|$. \square

Remark 5.8. Numerical methods produce the following estimates for these norms, each correct to 5 decimal places: $\|\mathbb{I} \bowtie \mathbb{N}\| \approx 1.24131$, $\|\mathbb{I} \bowtie \mathbb{I}\| \approx 1.25048$, $\|\mathbb{N} \mathbb{N} \mathbb{N}\| \approx 1.25655$ and $\|\mathbb{I} \mathbb{N} \mathbb{N}\| \approx 1.25906$. To see this, we apply the numerical algorithm described in [4] to $M(G)$ for each of these graphs G . The algorithm requires a unitary matrix without zero entries as a seed. Using the 4×4 Hadamard unitary $H_4 = H_2 \otimes H_2$ where $H_2 = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}$, after 20 or fewer iterations, in each case the algorithm produces real matrices R and S for which the Haagerup estimate gives an upper bound $\beta = c(S)c(R)$, and an orthogonal matrix U giving a lower bound $\alpha = \|M(G) \bullet U\|$, so that $\beta - \alpha < 10^{-6}$.

6. A CHARACTERISATION OF THE SCHUR IDEMPOTENTS WITH SMALL NORM

We now use the results of the previous section to characterise the Schur idempotents with norm η_k for $1 \leq k \leq 6$. This will yield a proof of Theorem 1.1.

Notation 6.1. We will write

$$\Gamma = \bigcup_{1 \leq m, n \leq \aleph_0} \Gamma(m, n).$$

Note that $\mathcal{N} = \{\|G\| : G \in \Gamma\} \setminus \{\infty\}$.

Remark 6.2. In the arguments below, we frequently encounter the following situation: G is a twin-free bipartite graph with an induced subgraph H , and H contains two vertices v_1 and v_2 which are twins (in H). Since G is twin-free, we can conclude that there is a vertex w

in G which is joined to one of v_1 and v_2 but not the other. We will say that the vertex w *distinguishes* the vertices v_1 and v_2 .

Lemma 6.3. *Let $G \in \Gamma$ be twin-free.*

- (1) *If $\deg(G) \geq 3$, then G contains either $\mathbb{I}\mathbb{N}$, $\mathbb{M}\mathbb{X}$ or $\mathbb{M}\mathbb{M}$ as an induced subgraph.*
- (2) *If $1 < \|G\| < \eta_4$, then $\deg(G) = 2$.*

Proof. (1) Let v be a vertex in G of degree at least 3 and consider an induced subgraph $\mathbb{A}\mathbb{N}$ with v at the top. Since G is twin-free, it is not hard to see that there are at least two other row vertices in G which distinguish the neighbours of v , and that this necessarily yields one of the induced subgraphs in the statement.

(2) follows from (1), since $\mathbb{I}\mathbb{N}$, $\mathbb{M}\mathbb{X}$ and $\mathbb{M}\mathbb{M}$ all have norm at least η_4 . \square

Lemma 6.4. *If $G \in \Gamma$ is connected with $\deg(G) = 2$ and $\|G\| < 4/\pi$, then $\|G\| = \|\Sigma(n, n)\|$ for some unique $n \geq 2$. Moreover,*

$$E \leq G \leq F$$

where $E = \Sigma(n, n)$ and

$$F = \begin{cases} \Sigma(n, n+1) & \text{if } n \text{ is even,} \\ \Lambda(n+1) & \text{if } n \text{ is odd.} \end{cases}$$

Proof. The graph G is connected and $\deg(G) = 2$, so G is either a path or a cycle. Since the sequence $\frac{2}{n} \cot \theta_n$ is strictly increasing with limit $4/\pi$ and $\frac{2}{n} \csc \theta_n > 4/\pi$ for every n , the claim follows from (*) and Theorem 4.2. \square

Lemma 6.5. *If $G \in \Gamma$ is twin-free with $\|G\| < \|\mathbb{M}\mathbb{X}\|$, then*

- (1) $G \not\geq \mathbb{M}\mathbb{N}$ and
- (2) $\deg(G) \leq 3$.

Proof. (1) Otherwise, since G is twin-free, there is a row vertex r in G which distinguishes the twin column vertices in $\mathbb{M}\mathbb{N}$. Hence either $G \geq \mathbb{M}\mathbb{X}$ or $G \geq \mathbb{M}\mathbb{M}$, and so $\|G\| \geq \|\mathbb{M}\mathbb{X}\|$ by Proposition 5.1.

(2) Suppose that $\deg(G) > 3$, so that $G \geq \mathbb{A}\mathbb{N}$. In order to distinguish between the four twin column vertices, there must be another row vertex in G attached to one but not all of these, so $\mathbb{I}\mathbb{A}\mathbb{N} \subseteq G$. In fact, to avoid the induced subgraph forbidden by (1), we must have $G \geq \mathbb{I}\mathbb{A}\mathbb{N}$. Distinguishing between the remaining columns using the same argument shows that $G \geq \mathbb{I}\mathbb{N}\mathbb{I}$, so $\|G\| \geq \|\mathbb{I}\mathbb{N}\mathbb{I}\| > \|\mathbb{M}\mathbb{X}\|$ by Proposition 5.4, contrary to hypothesis. \square

Notation 6.6. We define graphs $E_j \leq F_j$ for $1 \leq j \leq 6$ by:

$$\begin{aligned} E_1 &= F_1 = \mathbb{I}, & E_2 &= \mathbb{N}, & F_2 &= \mathbb{N}\mathbb{N}, & E_3 &= \mathbb{N}\mathbb{N}, & F_3 &= \mathbb{M}\mathbb{M}\mathbb{M} \\ E_4 &= \mathbb{I}\mathbb{N}, & F_4 &= \mathbb{A}\mathbb{N}\mathbb{N}, & E_5 &= F_5 = \mathbb{I}\mathbb{N}\mathbb{I}, & E_6 &= \mathbb{N}\mathbb{N}\mathbb{N}, & F_6 &= \mathbb{N}\mathbb{N}\mathbb{N}\mathbb{N}. \end{aligned}$$

Note that $\|E_j\| = \|F_j\| = \eta_j$ for $1 \leq j \leq 6$.

Theorem 6.7. *Let $G \in \Gamma$ be a twin-free, connected bipartite graph. For each $k \in \{1, 2, 3, 4, 5, 6\}$, the following are equivalent:*

- (1) $E_k \leq G \leq F_k$;
- (2) $\|G\| = \eta_k$;
- (3) $\eta_{k-1} < \|G\| \leq \eta_k$.

Proof. For each k , the implication (1) \implies (2) follows from Propositions 3.2 and 5.1, and (2) \implies (3) is trivial.

Suppose that G satisfies (3).

If $k = 1$, then $0 < \|G\| \leq 1$, so $\|G\| = 1$ and G is a disjoint union of complete bipartite graphs by [10, Theorem 4]. Since G is connected and twin-free, $G = \mathbb{I}$.

If $k \in \{2, 3\}$, then $\deg(G) = 2$ by Lemma 6.3, so $E_k \leq G \leq F_k$ by Lemma 6.4.

If $k \in \{4, 5, 6\}$ but $E_6 \neq G \neq F_6$, then $\deg(G) \neq 2$ by Lemma 6.4 and $\deg(G) \leq 3$ by Lemma 6.5, so $\deg(G) = 3$. Since $\|G\| < \|\mathbb{X}\mathbb{X}\| < \|\mathbb{X}\mathbb{X}\mathbb{X}\|$, we have $E_4 = \mathbb{I}\mathbb{N} \leq G$ by Lemma 6.3.

If G has the same row vertices as E_4 , then any column vertex c in G which is not in E_4 must be joined to E_4 so as to avoid the induced subgraph $\mathbb{X}\mathbb{X}$, and c cannot be joined to the degree 3 vertex of E_4 since $\deg(G) = 3$. Hence c must be joined to precisely one of the degree one row vertices of E_4 . Since G is twin-free, this gives $G \leq \mathbb{I}\mathbb{N}\mathbb{N} = F_4$, so $E_4 \leq G \leq F_4$.

If on the other hand G has at least four row vertices, choose a row vertex of G of smallest possible distance $\delta \in \{1, 2\}$ to the induced subgraph $E_4 \leq G$. If $\delta = 2$, then $\mathbb{I}\mathbb{N}\mathbb{N} \subseteq G$, and the rightmost row vertex r_4 of $\mathbb{I}\mathbb{N}\mathbb{N}$ is not connected to any of c_1, c_2, c_3 in G . Since $\mathbb{I}\mathbb{N}\mathbb{N}$ is not an induced subgraph of G by Proposition 5.5 and $\deg(G) = 3$, we have $G \geq \mathbb{I}\mathbb{N}\mathbb{N}$; but removing the two degree 1 vertices then shows that G contains the forbidden induced subgraph $\mathbb{X}\mathbb{X}$, a contradiction.

So $\delta = 1$. We claim that $G = E_5$. Indeed, since $\delta = 1$ we know that one of the following is an induced subgraph of G :

$$G_1 = \mathbb{I}\mathbb{N}\mathbb{N} \quad G_2 = E_5 = \mathbb{I}\mathbb{X}\mathbb{N} \quad G_3 = \mathbb{I}\mathbb{N}\mathbb{X} \quad G_4 = \mathbb{I}\mathbb{X}\mathbb{X} \quad G_5 = \mathbb{I}\mathbb{X}\mathbb{X}$$

Observe that $\mathbb{X}\mathbb{X}$ is an induced subgraph of both G_3 and G_4 , so the norms of these are too large. We can also rule out G_5 since it has a pair of twin row vertices of degree 3, so these cannot be distinguished in G . If $G_1 \leq G$, then since the vertices r_3 and r_4 are twins in G_1 but not in G , there is a column vertex c_4 attached to r_4 (say) but not r_3 . We cannot join c_4 to the maximal degree vertex r_2 , so we find that either

$$\mathbb{I}\mathbb{N}\mathbb{N} \quad \text{or} \quad \mathbb{I}\mathbb{X}\mathbb{N}$$

is an induced subgraph of G containing G_1 . However, the first is ruled out by Proposition 5.7 and the second contains an induced subgraph $\mathbb{X}\mathbb{X}$, so cannot occur either. So $E_5 \leq G$. If E_5 is a proper induced

subgraph of G , then since we must avoid \bowtie and also the induced subgraph \bowtie by Proposition 5.6, it follows that no column vertex of G has distance 1 to E_5 . So there is a row vertex of G with distance 1 to E_5 . Avoiding \bowtie and twin vertices of degree 3, we find an induced subgraph $\searrow \bowtie \swarrow \leq G$. To distinguish between the first two row vertices, we add a column vertex while avoiding \bowtie , and conclude that $\searrow \bowtie \swarrow \leq G$. Removing one row vertex gives $\searrow \bowtie \swarrow \leq G$, contradicting Proposition 5.7.

In summary: if $k = 4$, then $E_4 \leq G \leq F_4$; if $k = 5$, then $G = E_5$; and if $k = 6$ then $E_6 \leq G \leq F_6$. \square

Theorem 1.1 is an immediate consequence of Theorem 6.7 and Proposition 3.2. We also obtain:

Corollary 6.8. *Let $k \in \{1, 2, 3, 4, 5, 6\}$.*

(1) *If $G \in \Gamma$ is twin-free and connected, then*

$$\|G\| \leq \eta_k \iff G \leq F_j \text{ for some } j \leq k.$$

(2) *If $G \in \Gamma$, then $\|G\| = \eta_k$ if and only if:*

- (a) *each component H of G satisfies $\text{tf}(H) \leq F_j$ for some $j \leq k$; and*
- (b) *there is a component H of G with $E_k \leq \text{tf}(H)$.*

7. NORMAL MASA BIMODULE PROJECTIONS

Let \mathcal{H} be a separable Hilbert space \mathcal{H} . Given a masa (maximal abelian selfadjoint subalgebra) $\mathcal{D} \subseteq \mathcal{B}(\mathcal{H})$, we write $NCB_{\mathcal{D}}(\mathcal{B}(\mathcal{H}))$ for the set of normal completely bounded linear maps $\mathcal{B}(\mathcal{H}) \rightarrow \mathcal{B}(\mathcal{H})$ which are bimodular over \mathcal{D} . Smith's theorem [22] ensures that $\|\Phi\| = \|\Phi\|_{cb}$ for any $\Phi \in NCB_{\mathcal{D}}(\mathcal{B}(\mathcal{H}))$. Moreover, by [21, Theorem 2.3.7], there is a standard finite measure space (X, μ) so that \mathcal{D} is unitarily equivalent to $L^\infty(X, \mu)$ acting by multiplication on $L^2(X, \mu)$. Hence we will take $\mathcal{D} = L^\infty(X, \mu)$ and $\mathcal{H} = L^2(X, \mu)$ without loss of generality.

Recall that a set $R \subseteq X \times X$ is marginally null if $R \subseteq (N \times X) \cup (X \times N)$ for some null set $N \subseteq X$. Two Borel functions $\varphi, \psi: X \times X \rightarrow \mathbb{C}$ are equal marginally almost everywhere (m.a.e.) if $\{(x, y) \in X \times X: \varphi(x, y) \neq \psi(x, y)\}$ is marginally null. We write $[\varphi]$ for the equivalence class of all Borel functions which are equal m.a.e. to φ . Let $L^\infty(X, \ell^2)$ denote the Banach space of essentially bounded measurable functions $X \rightarrow \ell^2$, identified modulo equality almost everywhere. For $f, g \in L^\infty(X, \ell^2)$, we write $\langle f, g \rangle: X \times X \rightarrow \mathbb{C}$ for the function given m.a.e. by $\langle f, g \rangle(s, t) = \langle f(s), g(t) \rangle$. As shown in [10], there is a bijection

$$\Gamma: NCB_{\mathcal{D}}(\mathcal{B}(\mathcal{H})) \rightarrow \{[\langle f, g \rangle]: f, g \in L^\infty(X, \ell^2)\}$$

so that for every $\varphi \in \Gamma(\Phi)$, the map Φ is the normal extension to $\mathcal{B}(\mathcal{H})$ of pointwise multiplication by φ acting on the (integral kernels of)

Hilbert-Schmidt operators in $\mathcal{B}(\mathcal{H})$. Moreover, Γ is a homomorphism with respect to composition of maps and pointwise multiplication, and

$$\|\Phi\| = \inf\{\|f\| \|g\| : f, g \in L^\infty(X, \ell^2), \Gamma(\Phi) = [\langle f, g \rangle]\}$$

and this infimum is attained. In the discrete case, this reduces to [15, Corollary 8.8].

Lemma 7.1. *Let $\Phi \in NCB_{\mathcal{D}}(\mathcal{B}(\mathcal{H}))$. If $\Gamma(\Phi) = [\varphi]$ and $\{R_j\}_{j \geq 1}, \{C_j\}_{j \geq 1}$ are two countable Borel partitions of X with $\varphi^{-1}(\mathbb{C} \setminus \{0\}) \subseteq \bigcup_{j \geq 1} R_j \times C_j$, then $\|\Phi\| = \sup_j \|\Phi_j\|$ where $\Gamma(\Phi_j) = [\chi_{R_j \times C_j} \cdot \varphi]$.*

Proof. Let $P_j = \chi_{R_j}$ and $Q_j = \chi_{C_j}$. Note that $\{P_j\}$ and $\{Q_j\}$ are then partitions of the identity in \mathcal{D} . By [10, Theorem 10], the map Ψ given by $\Psi(T) = \sum_{j \geq 1} P_j T Q_j$ is in $NCB_{\mathcal{D}}(\mathcal{B}(\mathcal{H}))$, and

$$\Gamma(\Psi) = [\chi_K] \quad \text{where} \quad K = \bigcup_{j \geq 1} R_j \times C_j.$$

Since Γ is a homomorphism and $\varphi = \chi_K \cdot \varphi$, we have

$$\Gamma(\Phi) = \Gamma(\Psi) \cdot \Gamma(\Phi) = \Gamma(\Psi \circ \Phi),$$

hence $\Phi = \Psi \circ \Phi$. Let $\Psi_j \in NCB_{\mathcal{D}}(\mathcal{B}(\mathcal{H}))$ be given by $\Psi_j(T) = P_j T Q_j$, and let $\Phi_j = \Psi_j \circ \Phi$. Since Γ is a homomorphism, $\Gamma(\Phi_j) = \Gamma(\Psi_j \circ \Phi) = [\chi_{R_j \times C_j} \cdot \varphi]$, and for any $T \in \mathcal{B}(\mathcal{H})$,

$$\|\Phi(T)\| = \|\Psi \circ \Phi(T)\| = \sup_{j \geq 1} \|P_j \Phi(T) Q_j\| = \sup_{j \geq 1} \|\Phi_j(T)\|. \quad \square$$

Proposition 7.2. *Let $\Phi \in NCB_{\mathcal{D}}(\mathcal{B}(\mathcal{H}))$ be idempotent and let $\eta > \|\Phi\|$.*

- (1) *There exist a Borel set $G \subseteq X \times X$ and weakly Borel measurable functions $f, g: X \rightarrow \ell^2$ so that*
 - (a) $\Gamma(\Phi) = [\chi_G]$;
 - (b) $\chi_G(x, y) = \langle f(x), g(y) \rangle$ for all $x, y \in X$; and
 - (c) $\sup_{x, y \in X} \|f(x)\| \|g(y)\| < \eta$.
- (2) *For such a set G , there are two countable families of disjoint Borel subsets of X , say $\{R_j\}$ and $\{C_j\}$, so that the components of G are the Borel sets $G_j = G[R_j, C_j]$, and there are maps $\Phi_j \in NCB_{\mathcal{D}}(\mathcal{B}(\mathcal{H}))$ with $\Gamma(\Phi_j) = [\chi_{G_j}]$ and $\|\Phi\| = \sup_j \|\Phi_j\|$.*
- (3) *If F is a countable induced subgraph of G , then $\|F\| < \eta$.*
- (4) *If $\text{tf}(G)$ is countable, then $\|\Phi\| \leq \|\text{tf}(G)\|$.*

Proof. (1) We have $\Phi = \Phi \circ \Phi$, and Γ is a homomorphism. Hence if $\varphi: X \times X \rightarrow \mathbb{C}$ is Borel with $\Gamma(\Phi) = [\varphi]$, then $[\varphi] = \Gamma(\Phi) = \Gamma(\Phi)^2 = [\varphi^2]$, from which it follows that $[\varphi] = [\chi_G]$ where G is the Borel set $G = \varphi^{-1}(1)$. Hence there are $f, g \in L^\infty(X, \ell^2)$ with $[\chi_G] = [\langle f, g \rangle]$ and $\|f\| \|g\| = \|\Phi\| < \eta$; by multiplying f and g by $\chi_{X \setminus N}$ for some null set N and removing the marginally null set $(N \times X) \cup (X \times N)$ from G , we can achieve both pointwise equality $\chi_G = \langle f, g \rangle$ on $X \times X$ and $\sup_{x, y \in X} \|f(x)\| \|g(y)\| < \eta$.

- (2) As in [10], we can use the following argument of Arveson to show that G is a countable union of Borel rectangles. Since ℓ^2 is separable, the open set $\{(\xi, \eta) \in \ell^2 \times \ell^2 : \langle \xi, \eta \rangle \neq 0\}$ is a countable union $\bigcup_{n \geq 1} U_n \times V_n$ where U_n, V_n are open subsets of ℓ^2 . Let $A_n = f^{-1}(U_n)$ and $B_n = g^{-1}(V_n)$. These are Borel sets, and $G = \bigcup_{n \geq 1} A_n \times B_n$. Discard empty sets, so that $A_n, B_n \neq \emptyset$ for all $n \geq 1$.

For each $j \in \mathbb{N}$, the component of G containing A_j and B_j may be found as follows. Let $W_j^1 = \{j\}$, and for $k \geq 1$, let

$$W_j^{k+1} = \{n \in \mathbb{N} : \exists m \in W_j^k \text{ s.t. either } A_m \cap A_n \neq \emptyset \text{ or } B_m \cap B_n \neq \emptyset\}.$$

Let $W_j = \bigcup_{k \geq 1} W_j^k$, and consider the Borel sets $R_j = \bigcup_{n \in W_j} A_n$ and $C_j = \bigcup_{n \in W_j} B_n$. By construction, $G_j = G[R_j, C_j]$ is Borel. It is easy to see that G_j is the component of G containing A_j and B_j , and that every component of G is of this form for some j . Discard duplicates and relabel so that $G_j \neq G_k$ for $j \neq k$; the families $\{R_j\}$ and $\{C_j\}$ are then disjoint. Extending each family to a countable Borel partition of X and applying Lemma 7.1, we see that $\|\Phi\| = \sup_j \|\Phi_j\|$ where $\Phi_j = \Gamma^{-1}([\chi_{G_j}])$.

- (3) Let F be a countable induced subgraph of G , so that $F = G[A, B]$ for countable sets $A, B \subseteq X$. Considering the functions $f|_A \in \ell^\infty(A, \ell^2)$ and $g|_B \in \ell^\infty(B, \ell^2)$, we see that $\|F\| < \eta$ by [15, Corollary 8.8].
- (4) Now suppose that $F = \text{tf}(G) = G[A, B]$. By [15, Corollary 8.8], there are functions $f_A \in \ell^\infty(A, \ell^2)$ and $g_B \in \ell^\infty(B, \ell^2)$ so that

$$\langle f_A, g_B \rangle = \chi_{\text{tf}(G)} : A \times B \rightarrow \{0, 1\} \text{ and } \|f_A\| \|g_B\| = \|\text{tf}(G)\|.$$

For $x, y \in X$, write

$$G_x = \{y \in X : (x, y) \in G\} \quad \text{and} \quad G^y = \{x \in X : (x, y) \in G\}.$$

For each $a \in A$ and $b \in B$, the equivalence classes $S(a) = \{x \in X : G_a = G_x\}$ and $T(b) = \{y \in X : G^b = G^y\}$ are all Borel; indeed,

$$S(a) = f^{-1}(f(a) + \{g(y) : y \in Y\}^\perp)$$

and

$$T(b) = g^{-1}(g(b) + \{f(x) : x \in X\}^\perp).$$

Hence $\tilde{f} = \sum_{a \in A} f_A(a) \chi_{S(a)}$ and $\tilde{g} = \sum_{b \in B} g_B(b) \chi_{T(b)}$ are Borel functions $X \rightarrow \ell^2$, and $\chi_G(x, y) = \langle \tilde{f}(x), \tilde{g}(y) \rangle$ for every $x, y \in X$. So

$$\|\Phi\| \leq \|\tilde{f}\| \|\tilde{g}\| \leq \|f_A\| \|g_B\| = \|\text{tf}(G)\|. \quad \square$$

Corollary 7.3. *Let \mathcal{H} be a separable Hilbert space, and let \mathcal{D} be a masa in $\mathcal{B}(\mathcal{H})$. The set $\mathcal{N}(\mathcal{D}) = \{\|\Phi\| : \Phi \in \text{NCB}_{\mathcal{D}}(\mathcal{B}(\mathcal{H})), \Phi \text{ idempotent}\}$*

satisfies

$$\mathcal{N}(\mathcal{D}) \subseteq \{\eta_0, \eta_1, \eta_2, \eta_3, \eta_4, \eta_5\} \cup [\eta_6, \infty).$$

Proof. Let $k \in \{1, 2, 3, 4, 5, 6\}$ and suppose that $\Phi \in NCB_{\mathcal{D}}(\mathcal{B}(\mathcal{H}))$ is idempotent with $\eta_k > \|\Phi\|$. Taking $\eta = \eta_k$, let G, f, g, Φ_j be as in Proposition 7.2. Since $\|\Phi\| = \sup_j \|\Phi_j\|$, every Φ_j has $\|\Phi_j\| < \eta_k$. Hence we may assume that $\Phi = \Phi_1$, so that G is connected. Recall from §2 that $\mathcal{F}(G)$ is the set of (isomorphism classes of) finite, connected, twin-free subgraphs of G . If $F \in \mathcal{F}(G)$, then $\|F\| < \eta_k$ by Proposition 7.2(3), so $\|F\| \leq \eta_{k-1}$ by Theorem 1.1. By Corollary 6.8, $\mathcal{F}(G)$ consists entirely of induced subgraphs of some finite bipartite graph, so $\mathcal{F}(G)$ is finite. By Lemma 2.5, $\text{tf}(G) \in \mathcal{F}(G)$, so by Proposition 7.2(4), $\|\Phi\| \leq \|\text{tf}(G)\| \leq \eta_{k-1}$. \square

Question 7.4. Let \mathcal{H} be an infinite-dimensional separable Hilbert space. Do we have $\mathcal{N}(\mathcal{D}) = \mathcal{N}$ for every masa \mathcal{D} in $\mathcal{B}(\mathcal{H})$?

8. RANDOM SCHUR IDEMPOTENTS

For $p \in (0, 1)$ and $m, n \in \mathbb{N}$, let $\mathcal{G}(m, n, p)$ be the probability space of bipartite graphs in $\Gamma(m, n)$ where each of the possible mn edges appears independently with probability p .

Question 8.1. How does $\mathbb{E}_{m,n,p}(\|G\|)$, the expected value of the norm of the Schur idempotent arising from $G \in \mathcal{G}(m, n, p)$, behave as a function of m and n ?

Here is a crude result in this general direction.

Proposition 8.2. $\mathbb{E}_{m,n,p}(\|G\|) \rightarrow \infty$ as $\min\{m, n\} \rightarrow \infty$.

Proof. Let $s, t \in \mathbb{N}$, fix $H \in \Gamma(s, t)$ and let us write $\mathbb{P}_{m,n,p}(H \leq G)$ for the probability that a random graph $G \in \mathcal{G}(m, n, p)$ contains an induced subgraph isomorphic to H . We claim that

$$\mathbb{P}_{m,n,p}(H \leq G) \rightarrow 1 \quad \text{as } \min\{m, n\} \rightarrow \infty.$$

Indeed, as in [7, Proposition 11.3.1], one can see that the complementary event $H \not\leq G$ satisfies

$$\mathbb{P}_{m,n,p}(H \not\leq G) \leq (1 - r)^{\min\{\lfloor m/s \rfloor, \lfloor n/t \rfloor\}} \rightarrow 0 \quad \text{as } \min\{m, n\} \rightarrow \infty$$

where $r > 0$ is the probability that a random graph in $\mathcal{G}(s, t, p)$ is isomorphic to H . Hence

$$\begin{aligned} \mathbb{E}_{m,n,p}(\|G\|) &= \sum_{G \in \Gamma(m,n)} \|G\| \mathbb{P}_{m,n,p}(\{G\}) \\ &\geq \sum_{H \leq G \in \Gamma(m,n)} \|G\| \mathbb{P}_{m,n,p}(\{G\}) \\ &\geq \|H\| \sum_{H \leq G \in \Gamma(m,n)} \mathbb{P}_{m,n,p}(\{G\}) = \|H\| \mathbb{P}_{m,n,p}(H \leq G), \end{aligned}$$

so

$$\lim_{\min\{m,n\} \rightarrow \infty} \mathbb{E}_{m,n,p}(\|G\|) \geq \sup\{\|H\| : H \in \Gamma(s, t), s, t \in \mathbb{N}\} = \infty. \quad \square$$

For $p = 1/2$, we can say more about the growth rate of $\mathbb{E}_{m,n,p}(\|G\|)$.

Lemma 8.3. *Let $m, n \in \mathbb{N}$, fix an $m \times n$ matrix A with complex entries and let μ be the uniform probability measure on $M_{m,n}(\{-1, 1\})$. If*

$$\int_{\varepsilon \in M_{m,n}(\{-1, 1\})} \|\varepsilon \bullet A\|_{\bullet} d\mu(\varepsilon) = M,$$

then

$$\|\varepsilon \bullet A\|_{\bullet} \leq 4M$$

for every $\varepsilon \in M_{m,n}(\{-1, 1\})$.

Proof. Let ν be the probability measure on $M_{m,n}(\mathbb{T}) = \mathbb{T}^{m \times n}$ which is the product of $m \times n$ copies of normalised Haar measure on \mathbb{T} . The arguments in [9, §2.6] show that

$$\int_{z \in M_{m,n}(\mathbb{T})} \|\operatorname{Re}(z) \bullet A\|_{\bullet} d\nu(z) \leq M$$

and

$$\int_{z \in M_{m,n}(\mathbb{T})} \|\operatorname{Im}(z) \bullet A\|_{\bullet} d\nu(z) \leq M,$$

where $\operatorname{Re}(z) = [\operatorname{Re}(z_{ij})]$ and $\operatorname{Im}(z) = [\operatorname{Im}(z_{ij})]$. Hence

$$\int_{z \in M_{m,n}(\mathbb{T})} \|z \bullet A\|_{\bullet} d\nu(z) \leq 2M.$$

By [17, Theorem 2.2(i) and Remark 2.3], $A = B + C$ where $c(B) \leq 2M$ and $c(C^t) \leq 2M$. For any $\varepsilon \in M_{m,n}(\{-1, 1\})$, we have

$$\|\varepsilon \bullet B\|_{\bullet} \leq c(\varepsilon \bullet B) = c(B) \leq 2M$$

and similarly $\|\varepsilon \bullet C\|_{\bullet} \leq 2M$, so

$$\|\varepsilon \bullet A\|_{\bullet} \leq \|\varepsilon \bullet B\|_{\bullet} + \|\varepsilon \bullet C\|_{\bullet} \leq 4M. \quad \square$$

Proposition 8.4. $\mathbb{E}_{m,n,1/2}(\|G\|) \geq \frac{1}{8}\sqrt{\frac{k}{2}} - 1$ where $k = \min\{m, n\}$.

Proof. Let μ be the probability measure of the lemma, and write

$$M = \int \|\varepsilon\|_{\bullet} d\mu(\varepsilon).$$

Note that

$$\mathbb{E}_{m,n,1/2}(\|G\|) = \int \|2\varepsilon - \mathbf{1}\|_{\bullet} d\mu(\varepsilon) \geq 2M - 1,$$

where $\mathbf{1}$ is the all ones matrix. On the other hand, [6, Theorem 2.4] implies that there is a matrix $\varepsilon \in M_{m,n}(\{\pm 1\})$ with $\|\varepsilon\|_{\bullet} \geq \frac{1}{4}\sqrt{\frac{mn}{m+n}} \geq \frac{1}{4}\sqrt{\frac{k}{2}}$. By Lemma 8.3, $M \geq \frac{1}{4}\|\varepsilon\|_{\bullet}$. Combining these three inequalities gives the desired lower bound on $\mathbb{E}_{m,n,1/2}(\|G\|)$. \square

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